Vector Spaces and Subspaces

Vector Space:

A vector space \( \mathbb{V} \) is a nonempty collection of objects called vectors for which are defined the operations

- vector addition, denoted \( \bar{x} + \bar{y} \), and
- scalar multiplication (multiplication by a real constant), denoted \( c \bar{x} \),

that satisfy the following properties for all \( \bar{x}, \bar{y}, \bar{z} \in \mathbb{V} \) and \( c, d \in \mathbb{R} \).

Closure Properties:
1. \( \bar{x} + \bar{y} \in \mathbb{V} \).
2. \( c \bar{x} \in \mathbb{V} \).

Addition Properties:
3. There is a zero vector \( \bar{0} \) in \( \mathbb{V} \) such that \( \bar{x} + \bar{0} = \bar{x} \). \( \text{(Additive Identity)} \)
4. For every vector \( \bar{x} \in \mathbb{V} \), there is a vector \( -\bar{x} \) in \( \mathbb{V} \) (its negative) such that \( \bar{x} + (-\bar{x}) = \bar{0} \). \( \text{(Additive Inverse)} \)
5. \( (\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z}) \). \( \text{(Associativity)} \)
6. \( \bar{x} + \bar{y} = \bar{y} + \bar{x} \). \( \text{(Commutativity)} \)

Scalar Multiplication Properties:
7. \( 1\bar{x} = \bar{x} \). \( \text{(Scalar Multiplicative Identity)} \)
8. \( c(\bar{x} + \bar{y}) = c\bar{x} + c\bar{y} \). \( \text{(First Distributive Property)} \)
9. \( (c + d)\bar{x} = c\bar{x} + d\bar{x} \). \( \text{(Second Distributive Property)} \)
10. \( c(d\bar{x}) = (cd)\bar{x} \). \( \text{(Associativity)} \)

Vector Subspaces:

Vector Subspace Theorem
A nonempty subset \( \mathbb{W} \) of a vector space \( \mathbb{V} \) is a subspace of \( \mathbb{V} \) if it is closed under addition and scalar multiplication:

(i) If \( \bar{u}, \bar{v} \in \mathbb{W} \), then \( \bar{u} + \bar{v} \in \mathbb{W} \).
(ii) If \( \bar{u} \in \mathbb{W} \) and \( c \in \mathbb{R} \), then \( c\bar{u} \in \mathbb{W} \).

The Zero-Space Check:
The zero-space \( \{ \bar{0} \} \) is always a subspace of any vector space. If \( \bar{0} \) is not in \( \mathbb{W} \), then \( \mathbb{W} \) is empty and is not a subspace.
To be a subspace of $\mathbb{R}^3$, it must satisfy the following:

1. All multiples of $v$ are in $V$.
2. If $v_1, v_2 \in V$, then $v_1 + v_2 \in V$.
3. The zero vector $0$ is considered to be in $V$.

The only possible subspace of $\mathbb{R}^3$ are:
- A line through $0$.
- The plane $z = 0$.

### Vector Spaces

- **Definition of a Vector Space (Example 1)**
  - A set $V$ of objects (vectors) together with two operations: addition and scalar multiplication, such that:
    1. Closure under addition: If $u, v \in V$, then $u + v \in V$.
    2. Association law: $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$.
    3. Identity element of addition: There exists an element $0 \in V$ such that $u + 0 = u$ for all $u \in V$.
    4. Inverse element of addition: For each $u \in V$, there exists an element $-u \in V$ such that $u + (-u) = 0$.
    5. Identity element of scalar multiplication: $1u = u$ for all $u \in V$.
    6. Compatibility of scalar multiplication with field multiplication: $c(u + v) = cu + cv$ for all $c \in \mathbb{R}$ and $u, v \in V$.
    7. Distributivity of scalar multiplication over vector addition: $(c + d)u = cu + du$ and $c(u + v) = cu + cv$ for all $c, d \in \mathbb{R}$ and $u, v \in V$.

**Example 2**: Consider $\mathbb{R}^2$ with the usual definition of vector addition and scalar multiplication. Is $\mathbb{R}^2$ a vector space?

- **Solution**
  - Yes, $\mathbb{R}^2$ satisfies all the properties of a vector space.

- **Subspaces**
  - A subspace of a vector space is a subset that is itself a vector space.
  - Example: $\mathbb{R}^2$ itself is a subspace of $\mathbb{R}^2$.

**Some Problems**

1. Find the possible subspaces of $\mathbb{R}^2$.
2. Prove that the set $S$ of all vectors $x$ such that $x \cdot x = 0$ is a subspace of $\mathbb{R}^2$.

**Span of a Set of Vectors**

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in $\mathbb{R}^n$. The **span** of $S$, denoted as $\text{span}(S)$, is the set of all linear combinations of vectors in $S$. That is, if $a_1, a_2, \ldots, a_n$ are scalars, then $\text{span}(S) = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n | a_i \in \mathbb{R}\}$.

**Example**: Find the span of the set $S = \{v_1, v_2\}$ where $v_1 = (1, 0)$ and $v_2 = (0, 1)$.

- **Solution**
  - The span of $S$ is $\mathbb{R}^2$ itself, since any vector in $\mathbb{R}^2$ can be written as a linear combination of $v_1$ and $v_2$.

**Immediate Results**

- Any vector $x$ can be written as a linear combination of any vector $x$.
- Any vector $x$ can be scaled by any scalar $c$.
- The zero vector $0$ is in $\text{span}(S)$.
- If $x \in \text{span}(S)$, then $-x \in \text{span}(S)$.
- If $x_1, x_2 \in \text{span}(S)$, then $x_1 + x_2 \in \text{span}(S)$.
- If $x \in \text{span}(S)$ and $c \in \mathbb{R}$, then $cx \in \text{span}(S)$.