Complex Solutions of the Eigenproblem

Problems with real eigenvalues are the primary focus in Math 4A. However, sometimes the case of complex eigenvalues and eigenvectors is taught. This will be essential when applied in Math 4B.

First a quick review of complex arithmetic.

One way to write a complex number \( z \) is

\[
z = x + iy, \quad \text{where} \quad i = \sqrt{-1} \quad (i^2 = -1)
\]

and \( x = \Re z \), the real part of \( z \)

\( y = \Im z \), the imaginary part of \( z \)

(Note that \( x \) and \( y \) are both real numbers.)

The complex conjugate of \( z \) (here denoted as \( z^* \)) is

\[
z^* = x - iy, \quad \text{ie change the sign of the imaginary part of} \ z.
\]

Given \( z_1 = x_1 + iy_1 \) and \( z_2 = x_2 + iy_2 \)

Then:

Add \( z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \)

Subtract \( z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \)

Multiply \( z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) \)

\[
= x_1x_2 + iy_1x_1 + ix_2y_1 + i^2y_1y_2
= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)
\]

Divide \( \frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{z_2^*}{z_2^*} \)

\[
= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}
= \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2} + i \left( \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \right)
\]
When computation is extended into complex numbers, all square matrices may be solved for their eigenvalues and eigenvectors.

Example problem: Find eigenvalues and eigenvectors for \( A = \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix} \)

\[
\det(A - \lambda I) = 0 \implies \det \begin{bmatrix} 4 - \lambda & -2 \\ 1 & 6 - \lambda \end{bmatrix} = 0 \implies \lambda^2 - 10\lambda + 26 = 0
\]

\[
\lambda = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 26}}{2} = \frac{10 \pm \sqrt{-4}}{2} = 5 \pm i
\]

Two eigenvalues, \( \lambda_1 = 5 + i \), \( \lambda_2 = 5 - i \)

The eigenvalues always are conjugate pairs when complex.

Now find eigenvectors which satisfy \( (A - \lambda I) \vec{v} = \vec{0} \)

For \( \lambda = \lambda_1 = 5 + i \)

\[
\begin{bmatrix} 4 - \lambda & -2 \\ 1 & 6 - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 4 - (5 + i) & -2 \\ 1 & 6 - (5 + i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -1 - i & -2 \\ 1 & 1 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

This says \((-1 - i) v_1 - 2 v_2 = 0\) and \(1 v_1 + (1 - i) v_2 = 0\)

Just as for the case of real eigenvalues, when working with 2x2 matrices, the two rows say the same thing about the components of the eigenvector \( \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \).

Let's see clearly that the two equations are just multiples of each other.

The first equation is \((-1 - i)\) times the second equation.

\((-1 - i) [v_1 + (1-i) v_2 = 0] = (-1-i) v_1 + (-1-i)(1-i) v_2 = (-1-i) v_1 - 2 v_2 = 0\)
As always, for matrix $A$ of dimension $3 \times 3$ and greater, you must solve the homogeneous equation as normal, i.e. do RREF, identify free variable(s), and etc. The eigenvector(s) are the basis for the null space of $A - \lambda I$.

So, returning to the example problem, the first equation gives $v_2 = -1 - i \, v_1$ and thus

$$v_1 = \begin{bmatrix} 1 \\ i \\ -1 - i \end{bmatrix}$$

This is typically scaled to eliminate the fraction, i.e.

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 - i \end{bmatrix}$$

If we had worked with the second equation, i.e.

$$1 v_1 + (1 - i) v_2 = 0$$

we obtain $v_1 = \begin{bmatrix} 1+i \\ i \\ i \end{bmatrix}$.

These two $v_1$'s are just multiples of each other (obscured by complex numbers).

Then using $\lambda = \lambda_2 = 5 - i$, you obtain $v_2 = \begin{bmatrix} 2 \\ -1 + i \end{bmatrix}$.

Eigenvalues which are conjugate pairs have associated eigenvectors which are conjugate pairs.

The recommended approach is to use $\lambda_1 = a + ib$, solve for $v_1$, and then for $\lambda_2 = a - ib$ immediately write $v_2$ as $v_2 = v_1^* \; \text{and not explicitly solve for} \; v_2$. 
Matrices with complex eigenvalues/eigenvectors do not diagonalize, but may be placed in a standard format. The following illustrates the case for a 2x2 matrix, A. For $\lambda = a + ib$ and eigenvector $\vec{v}_i$, the standard form is

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $C$ is related to $A$ by

$$C = P^{-1}AP$$

using

$$P = \begin{bmatrix} \text{Im} \vec{v}_i & \text{Re} \vec{v}_i \end{bmatrix}$$

Back to our numerical example:

$\lambda = 5 + i$  \quad $\vec{v}_i = \begin{bmatrix} 2 \\ -1-i \end{bmatrix}$  \quad So $a = 5$ and $b = 1$

$\text{Re} \vec{v}_i = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$  \quad $\text{Im} \vec{v}_i = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

So $P = \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix}$  \quad $P^{-1} = \frac{1}{2} \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$

$C = \frac{1}{2} \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -6 & -10 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix}$

$$= \frac{1}{2} \begin{bmatrix} 10 & -2 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} a-b \\ b & a \end{bmatrix}$$

The $C$ is a combination of a scaling by $r = \sqrt{a^2 + b^2}$ and a rotation by $\Theta$, i.e.,

$$C = r \begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = r \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix}$$

The example has $r = \sqrt{5^2 + 1^2} = \sqrt{26}$ and $\sin \Theta = \frac{1}{\sqrt{26}}$  \quad $\Theta = 11.3^\circ$