Projectile Motion

Projectile motion is an idealized description for the motion of a point mass that has been given an initial velocity and then follows a path. Idealizations include:

- The only force acting on the mass is the force due to gravity. (No air resistance)
- The curvature and rotation of the earth are ignored. (Motion is close to earth's surface and not of long distance.)

We start with our now familiar one dimensional motion equations for a constant acceleration and use them to obtain them to describe motion in the x direction and the y direction. Let the x coordinate axis be horizontal and the y coordinate axis be vertical with positive up. Also, let the origin of the coordinate system be at the point of initial velocity. Then in the x direction we have:

\[ a_x = 0 \quad \text{(Always)} \]
\[ v_x = v_{0x} \quad 1.) \]
\[ x = v_{0x} t \quad 2.) \]

and in the y direction we have:

\[ a_y = -g \]
\[ v_y = -gt + v_{0y} \quad 3.) \]
\[ y = -\frac{gt^2}{2} + v_{0y} t \quad 4.) \]
\[ y = \left( \frac{v_y + v_{oy}}{2} \right) t \quad \text{(5)} \]

\[ v_y^2 = v_{oy}^2 - 2gy \quad \text{(6)} \]

Notice that the choice of x-y origin results in \( x_0 = 0 \) and \( y_0 = 0 \). Confirm for yourself that the last two equations have no useful counterpart for motion in the x direction for which \( a_x = 0 \).

Often, but not always, the initial velocity is specified. It may be given in either of two ways:

\[ y \]

\[ V_0 = |\overrightarrow{V_0}| \]

\[ \theta_0 \]

\[ x \]

\[ v_{ox} \]

\[ v_{oy} \]

or

These are related as:

\[ v_{ox} = V_0 \cos \theta_0, \quad v_{oy} = V_0 \sin \theta_0 \]

\[ V_0 = \sqrt{v_{ox}^2 + v_{oy}^2}, \quad \theta_0 = \tan^{-1} \left( \frac{v_{oy}}{v_{ox}} \right) \]

The kinematics of the x motion and the y motion are separate but the fact that they share a common time \( t \) makes it possible (and necessary) to switch between x equations and y equations in solving specific problems.
When the initial velocity has $v_{oy} > 0$, two useful results are available. The vertical velocity becomes zero at the projectile's maximum height, thus equation 6 yields

$$y_{\text{max}} = \frac{v_{oy}^2}{2g}$$

The $x$ distance traveled when the projectile returns to its initial height of $y = 0$ is called the range. (Strictly a horizontal distance.) Equation 4 solves for the time of this event

$$t = \frac{2v_{oy}}{g}$$

and then equation 2 gives

$$\text{Range} = \frac{2v_{oy}v_{ox}}{g}$$

This may be written as

$$\text{Range} = \frac{v_o^2 \sin(2\theta_o)}{g}$$

(Using $\sin(2\theta_o) = 2\sin\theta_o \cos\theta_o$)
Example Problem: A ball rolls off a horizontal table 0.75 m above the floor. It hits the floor a horizontal distance of 1.40 m from the edge of the table. Find: a) time t to hit the floor, b) magnitude of initial velocity \( V_0 \), and c) magnitude and direction of velocity just before hitting the floor.

We note that \( \theta_0 = 0 \)

\[ V_{0x} = V_0 \]
\[ V_{0y} = 0 \]

\[ y = -\frac{gt^2}{2} + V_{0y}t \Rightarrow -0.75m = -9.8m/s^2 \frac{t^2}{2} + 0.1t \]

Note that the floor is at \( y = -0.75 \) m in x-y system.

\[ t = \sqrt{\frac{2(0.75m)}{9.8m/s^2}} = \sqrt{0.153s^2} = 0.39s \]

Eqn 2) with this time gives

\[ x = V_{0x}t \Rightarrow V_{0x} = \frac{1.40m}{0.39s} = 3.6m/s \]

But \( V_{0x} = V_0 \) so \( V_0 = 3.6m/s \)

Eqn 3) at \( t = 0.39s \) gives

\[ V_y = -gt + V_{0y} = -9.8m/s^2 \times 0.39s + 0 = -3.8m/s \]

Thus \[ |\mathbf{v}| = \sqrt{V_{x}^2 + V_y^2} = \sqrt{(3.6m/s)^2 + (-3.8m/s)^2} = 5.2m/s \]

\[ \theta = \tan^{-1}\left|\frac{-3.8}{3.6}\right| = \tan^{-1}(1.06) = 47^\circ \]

Below horizontal