Decomposition of a Given Vector

Problem: Given vector \( \bar{v} = (1, -9, 1) \in \mathbb{R}^3 \), write \( \bar{v} \) as \( \bar{w} + \bar{w}_\bot \) where \( \bar{w} \) is in the subspace \( (a, b, a - 2b) \) of \( \mathbb{R}^3 \) and \( \bar{w}_\bot \) is perpendicular to the subspace \( (a, b, a - 2b) \).

Defining a subspace in this manner is saying that the subspace consists of all vectors in \( \mathbb{R}^3 \) which may be written as \( a(1, 0, 1) + b(0, 1, -2) \) that is, \( \bar{d}_1 = (1, 0, 1) \) and \( \bar{d}_2 = (0, 1, -2) \) are basis vectors for the given subspace. However \( \bar{d}_1 \) and \( \bar{d}_2 \) are neither orthogonal nor unit vectors, since
\[
\bar{d}_1 \cdot \bar{d}_2 = -2 \neq 0
\]
\[
\| \bar{d}_1 \| = \sqrt{1^2 + 1^2} = \sqrt{2} \neq 1
\]
\[
\| \bar{d}_2 \| = \sqrt{1^2 + (-2)^2} = \sqrt{5} \neq 1
\]

Let \( \bar{u}_1 \) and \( \bar{u}_2 \) be orthonormal basis vectors for the same subspace (ie both orthogonal and unit vectors). The first, \( \bar{u}_1 \), may be taken as just \( \bar{d}_1 \) made a unit vector, ie
\[
\bar{u}_1 = \frac{\bar{d}_1}{\| \bar{d}_1 \|} = \frac{(1, 0, 1)}{\sqrt{2}} = \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)
\]
To find the \( \overline{u}_z \), we first find the vector component of \( \overline{d}_z \) in the direction of \( \overline{u}_1 \), i.e.

\[
(\overline{d}_z \cdot \overline{u}_1) \overline{u}_1
= \left[ (0, 1, -2) \cdot \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right] \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)
= -\frac{2}{\sqrt{2}} \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = (-1, 0, -1)
\]

This is subtracted from \( \overline{d}_z \), i.e.

\[
(0, 1, -2) - (-1, 0, -1) = (1, 1, -1)
\]

and then this is made a unit vector to become \( \overline{u}_2 \), i.e.

\[
\overline{u}_2 = \frac{(1, 1, -1)}{\| (1, 1, -1) \|} = \frac{(1, 1, -1)}{\sqrt{3}}
\]

\[
\overline{u}_2 = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)
\]

Clearly \( \overline{u}_1 \) and \( \overline{u}_2 \) are unit vectors, and since \( \overline{u}_1 \cdot \overline{u}_2 = 0 \), they are orthogonal.

Also \( \overline{d}_1 = \sqrt{2} \overline{u}_1 \),

and \( \overline{d}_2 = -\sqrt{2} (\overline{u}_1) + \sqrt{3} (\overline{u}_2) \)

So \( \overline{d}_1 \) with \( \overline{d}_2 \) span the same subspace as do \( \overline{u}_1 \) with \( \overline{u}_2 \), i.e.

\[
\text{Span} \left\{ \overline{d}_1, \overline{d}_2 \right\} = \text{Span} \left\{ \overline{u}_1, \overline{u}_2 \right\}
\]
The desired vector $\overline{w}$, i.e. the portion of $\overline{u}$ in the given subspace is then:

$$\overline{w} = (\overline{v} \cdot \overline{u}_1) \overline{u}_1 + (\overline{v} \cdot \overline{u}_2) \overline{u}_2$$

$$= (1, -9, 1) \cdot \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$+ (1, -9, 1) \cdot \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$$

$$= \frac{2}{\sqrt{2}} \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) + \frac{-9}{\sqrt{3}} \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$$

$$= (1, 0, 1) + (-3, -3, 3)$$

$$\overline{w} = (-2, -3, 4)$$

and $\overline{w}_\perp$ is what remains of $\overline{u}$ when $\overline{w}$ is subtracted, i.e.

$$\overline{w}_\perp = \overline{v} - \overline{w} = (1, -9, 1) - (-2, -3, 4)$$

$$\overline{w}_\perp = (3, -6, -3)$$

Let's check our result.

Clearly $\overline{v} = \overline{w} + \overline{w}_\perp$

and $\overline{w}_\perp$ is not in the given subspace since

$$\overline{w}_\perp \cdot \overline{d}_1 = (3, -6, -3) \cdot (1, 0, 1) = 0$$

and $$\overline{w}_\perp \cdot \overline{d}_2 = (3, -6, -3) \cdot (0, 1, -2) = 0$$