Orthogonal Projection

**Theorem 8**

The Orthogonal Decomposition Theorem

Let $W$ be a subspace of $\mathbb{R}^n$. Then each $y$ in $\mathbb{R}^n$ can be written uniquely in the form

$$y = \hat{y} + z$$

(1)

where $\hat{y}$ is in $W$ and $z$ is in $W^\perp$. In fact, if $\{u_1, \ldots, u_p\}$ is any orthogonal basis of $W$, then

$$\text{proj}_W y = \hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

(2)

and $z = y - \hat{y}$.

The vector $\hat{y}$ in (1) is called the **orthogonal projection of $y$ onto $W$** and often is written as $\text{proj}_W y$. See Fig. 2. When $W$ is a one-dimensional subspace, the formula for $\hat{y}$ matches the formula given in Section 6.2.

**Figure 2** The orthogonal projection of $y$ onto $W$.

**Example 2**

Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Observe that $\{u_1, u_2\}$ is an orthogonal basis for $W = \text{Span}\{u_1, u_2\}$. Write $y$ as the sum of a vector in $W$ and a vector orthogonal to $W$.

**Solution**

The orthogonal projection of $y$ onto $W$ is

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{15}{30} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$$

Also

$$y - \hat{y} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$$

Theorem 8 ensures that $y - \hat{y}$ is in $W^\perp$. To check the calculations, however, it is a good idea to verify that $y - \hat{y}$ is orthogonal to both $u_1$ and $u_2$ and hence to all of $W$. The desired decomposition of $y$ is

$$y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix} \quad \blacksquare$$
Let's see how this works for some obvious cases.

Let \( \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and \( \vec{u} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \)

\[
\text{proj}_\vec{u} \vec{y} = \hat{y} = \frac{\vec{y} \cdot \vec{e}_2}{\vec{e}_2 \cdot \vec{e}_2} \vec{e}_2 = \frac{2}{1} \vec{e}_2 = 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}
\]

We get the vector part of \( \vec{y} \) in the \( \vec{u} = \vec{e}_2 \) direction. In this example \( W = \text{Span} \{ \vec{e}_2 \} \) and it is common to write \( \text{proj}_\vec{u} \vec{y} \) instead of \( \text{proj}_W \vec{y} \) when \( W \) is the span of a single vector.

What if \( \vec{u} = 2 \vec{e}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \)?

Then \( \text{proj}_\vec{u} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{2}{4} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \)

We can think of \( \frac{\text{Scalar amount of } \vec{y} \text{ in } \vec{u} \text{ direction}}{\text{Direction information}} \)

\[
\text{proj}_\vec{u} \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \quad \text{as } \quad \left( \frac{\vec{y} \cdot \vec{u}}{||\vec{u}||} \right) \frac{\vec{u}}{||\vec{u}||}
\]

This is since \( ||\vec{u}|| = \sqrt{\vec{u} \cdot \vec{u}} \) and recalling that \( \frac{\vec{u}}{||\vec{u}||} \) makes a unit vector version of \( \vec{u} \).

So, \( \frac{\vec{y} \cdot \vec{u}}{||\vec{u}||} \) is a signed scalar giving the amount of \( \vec{y} \) in the vector \( \vec{u} \) direction and \( \frac{\vec{u}}{||\vec{u}||} \) is a vector giving the direction of \( \vec{u} \) but of unit length.

Note: \( \vec{y} = \hat{y} + \vec{z} \), \( \hat{y} \) (the projection) is in subspace \( W \) and \( \vec{z} \), i.e. \( \vec{y} - \hat{y} \), is orthogonal to \( W \) and thus in \( W^\perp \).

\( \hat{y} \) is the closest point in \( W \) to \( \vec{y} \) and thus the best approximation to \( \vec{y} \) in the subspace \( W \).

The length \( ||\vec{z}|| \) is the minimum distance from the point \( \vec{y} \) to the subspace \( W \).
If the given vectors spanning \( \mathbf{W} \) are not orthogonal, the following process may be used if \( \mathbf{W} \) is the span of only two vectors \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \), i.e. \( \mathbf{W} = \text{Span} \{ \mathbf{u}_1, \mathbf{u}_2 \} \) to find vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) which are orthogonal, i.e. \( \mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \), and still have the same subspace \( \mathbf{W} \), i.e. \( \mathbf{W} = \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2 \} \).

To find \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \):

Let \( \mathbf{v}_1 = \mathbf{u}_1 \)

\[
\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2
\]

Example: \( \mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \mathbf{u}_1 \cdot \mathbf{u}_2 = 3 + 8 + 3 = 14 \neq 0 \)

Thus \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are not orthogonal, but are a basis for \( \mathbf{W} = \text{Span} \{ \mathbf{u}_1, \mathbf{u}_2 \} \).

Let \( \mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \)

Form \( \text{proj}_{\mathbf{v}_1} \mathbf{u}_2 = \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \frac{(3+8+3)}{14} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \)

Then \( \mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{u}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)

Check: \( \mathbf{v}_1 \cdot \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 + 2 - 3 = 0 \), so orthogonal

Easily see that \( \mathbf{v}_1, \mathbf{v}_2 \in \text{Span} \{ \mathbf{u}_1, \mathbf{u}_2 \} \)

and \( \mathbf{u}_1, \mathbf{u}_2 \in \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2 \} \)

So \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is an orthogonal bases set of the same subspace \( \mathbf{W} \).
Student: For $\mathbf{y} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Find $\text{proj}_{\mathbf{u}_1} \mathbf{y}$  

Answer: $\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$

Then, let $W = \text{Span} \left\{ \mathbf{u}_1, \mathbf{u}_2 \right\}$

Find $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$  

Answer $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

What is $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$?  

Answer $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Given basis set $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

Are the linear independent?  

Are these orthogonal?  

Find an orthogonal basis set based on these.

Answer $\begin{bmatrix} \frac{2}{3} \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$