4.1 VECTOR SPACES AND SUBSPACES

Much of the theory in Chapters 1 and 2 rested on certain simple and obvious algebraic properties of $\mathbb{R}^n$, listed in Section 1.3. In fact, many other mathematical systems have the same properties. The specific properties of interest are listed in the following definition.

**Definition**

A vector space is a nonempty set $V$ of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $u, v, w$ in $V$ and for all scalars $c$ and $d$.

1. The sum of $u$ and $v$, denoted by $u + v$, is in $V$.
2. $u + v = v + u$.
3. $(u + v) + w = u + (v + w)$.
4. There is a zero vector $0$ in $V$ such that $u + 0 = u$.
5. For each $u$ in $V$, there is a vector $-u$ in $V$ such that $u + (-u) = 0$.
6. The scalar multiple of $u$ by $c$, denoted by $cu$, is in $V$.
7. $c(u + v) = cu + cv$.
8. $(c + d)u = cu + du$.
9. $c(du) = (cd)u$.
10. $1u = u$.

**Definition**

A subspace of a vector space $V$ is a subset $H$ of $V$ that has three properties:

a. The zero vector of $V$ is in $H$.

b. $H$ is closed under vector addition. That is, for each $u$ and $v$ in $H$, the sum $u + v$ is in $H$.

c. $H$ is closed under multiplication by scalars. That is, for each $u$ in $H$ and each scalar $c$, the vector $cu$ is in $H$. 