Minimizing the distance from a point to a line in $\mathbb{R}^3$ : 4 different Methods

The Problem:

Given a point $P = (x, y, z)$ and a line $\ell(t) = P_0 + \hat{v} \cdot t = (x_0 + at, y_0 + bt, z_0 + ct)$, find the minimum distance from the point to the line.

Note: $P_0 = (x_0, y_0, z_0)$ is a known point on the line, $\hat{v} = (a, b, c)$ is a vector parallel to the line.

Method 1: Vector projection

- Find the vector from $P_0$ to $P$: $\vec{PP}_1 = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$

- Find the vector projection of $\vec{PP}_1$ onto $\hat{v}$, call it $\vec{a}$

\[
\vec{a} = \frac{(\vec{PP}_1 \cdot \hat{v})}{\|\hat{v}\|^2} \hat{v}
\]
* Subtract \( \vec{a} \) from \( \vec{PP}_1 \) to get the vector from the point on the line nearest \( P_i \) to \( P_1 \)

\[
\begin{align*}
\vec{a} &= \frac{\vec{PP}_1 \cdot \vec{V}}{||\vec{V}||^2} \\
\text{where } \vec{a} &= \left( \frac{\vec{PP}_1 \cdot \vec{V}}{||\vec{V}||^2} \right) \vec{V}
\end{align*}
\]

* The magnitude of this vector is our minimum distance

\[
\text{min} = ||\vec{P_0P}_1 - \vec{a}||
\]

Method 2: Scalar projection

* Find the vector from \( P_0 \) to \( P_1 \): \( \vec{PP}_1 = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \)

* Find the scalar projection of \( \vec{PP}_1 \) onto \( \vec{V} \), call it \( a \)

\[
a = \frac{\vec{PP}_1 \cdot \vec{V}}{||\vec{V}||}
\]
We now have a right triangle with adjacent side length \( a \), opposite side length \( d_{\text{min}} \), and hypotenuse \( ||\overrightarrow{p_i}|| \)

By the Pythagorean Theorem:

\[
d_{\text{min}} = \sqrt{||\overrightarrow{p_i}||^2 - a^2}
\]

where \( a = \frac{\overrightarrow{p_i} \cdot \overrightarrow{v}}{||\overrightarrow{v}||} \)

**Method #3 using the cross product**

- Find the vector from \( P_0 \) to \( P_i \): \( \overrightarrow{P_0P_i} = (x_i - x_0, y_i - y_0, z_i - z_0) \)

If \( \phi \) is the angle between \( \overrightarrow{P_0P_i} \) and \( \overrightarrow{v} \),

\[
d_{\text{min}} = ||\overrightarrow{P_0P_i}|| \sin \phi = \frac{||\overrightarrow{P_0P_i} \times \overrightarrow{v}||}{||\overrightarrow{v}||}
\]
Method #4  Using Calculus

- The distance between any point in $\mathbb{R}^3$ $(x, y, z)$ and $P_1$ is
  \[ d = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} \]

- Let $f(x, y, z)$ be the distance squared from any point in $\mathbb{R}^3$ to $P_1$:
  \[ f(x, y, z) = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 \]

- To find the distance squared from any point on the line $\mathbf{l}(t) = (x_0 + at, y_0 + bt, z_0 + ct)$ to $P_1$, substitute $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$ into $f$:
  \[ f(t) = (x_0 + at - x_1)^2 + (y_0 + bt - y_1)^2 + (z_0 + ct - z_1)^2 \]

- To find the value of $t$ that minimizes $f$, differentiate $f(t)$, set $f'(t) = 0$, and solve for $t$:
  \[ f'(t) = 2a(x_0 + at - x_1) + 2b(y_0 + bt - y_1) + 2c(z_0 + ct - z_1) = 0 \]

Solving for $t$ gives:
\[ t = \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{a^2 + b^2 + c^2} \]

- Substitute this value of $t$ into $f$ and take the square root to determine the minimum distance.